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INEXTENSIONAL BENDING OF A SHELL TRIANGULAR ELEMENT IN QUADRATIC PARAMETRIC REPRESENTATION†

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Abstract—Exact closed form inextensional bending solutions are developed for a class of shell triangular elements in quadratic parametric representation; these elements may have positive, zero or negative Gaussian curvature. Previous exact closed form inextensional solutions of the equations of thin shell theory are known, for non-zero Gaussian curvature, only for spherical surfaces.

The polynomial solutions are relevant to finite element design and provide inextensional bending behaviours with slowly varying curvature changes ready for use in patch test validation of any shell finite element.

1. INTRODUCTION

A study of the equations of first approximation shell theory [1–3] reveals that the theory admits four characteristic solution modes which correspond essentially to rigid body movements, momentless membrane stresses, inextensional bending and edge effects. In well designed shells, the membrane mode usually produces the dominant fibre stresses with inextensional bending making significant contributions only to the deformation state. Edge effects decay rapidly away from discontinuities and it is customary to consider them in isolation from the other modes. Of the first mentioned modes, it is the problem of inextensional bending which has provided one of the most difficult obstacles in developing a wholly satisfactory curved finite element within first approximation shell theory [4, 5].

Closed form exact solutions for the displacements of inextensional bending in shell surfaces with non-zero Gaussian curvature are known (to this author) only for spherical surfaces [6]. Thus, a primary objective is to develop such solutions covering a much wider range of surfaces which may have positive, zero or negative Gaussian curvature. The polynomial form adopted for the solutions is relevant to the techniques of finite element analysis and provides inextensional bending behaviours ready for use in patch test [4] validation of any shell finite element. Indeed, a Fortran computer program which calculates displacements, rotations and curvature changes from the cubic polynomial solutions is described elsewhere [7].

The technique of polynomial parametric representation of the rectangular cartesian coordinates is widely employed in the finite element method to describe curved surfaces and/or boundaries. When the rectangular cartesian components of displacement are also parametrically represented by the same polynomials then the representation is called isoparametric [8, 9]. While isoparametric representation admits exact recovery of the rigid body movements, it does not provide an acceptable description of the bending of curved surfaces because its bending is accompanied by middle surface strains which have significant magnitude and cannot be ignored. In contrast, the bending solutions presented in the sequel correspond to zero middle surface strain. They are derived by extending the polynomial degree of displacement representation for a class of triangular elements with doubly curved shell surfaces which are parametrically represented in terms of quadratic polynomials of the surface coordinates. The element class is characterised by the planes of the quadratic arcs which define the sides, these planes intersect along three parallel lines. The present purpose is adequately served by assuming that these parallel lines are normal to the plane which passes through the vertices of the triangular element. This assumption offers presentational simplification but is not essential to the method and form of solution.

The analysis is embedded in the strain/displacement and curvature change/displacement

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equations of first approximation shell theory[1–3] and it is convenient to commence with a statement of the relevant vector equations in terms of an arbitrary orthogonal system of curvilinear surface coordinates. These equations form the basis for evaluation of the physical quantities in any chosen direction and are then related to the equations of the oblique curvilinear coordinate system belonging to the parametric representation. Particular equations for the shell triangular element are derived prior to a study of the quality of the physical quantities when they are described isoparametrically. Closed form exact polynomial solutions to the inextensional bending problem are then presented with specimen numerical values; the cubic polynomial solutions correspond to curvature changes which, although not generally constant, are slowly varying for the shallow shell elements usually encountered in finite element analysis.

2. GEOMETRY AND KINEMATICS IN ORTHOGONAL CURVILINEAR COORDINATES

Let X, Y, Z refer to a fixed right handed orthogonal cartesian coordinate system and denote by vector r the position of a point on the middle surface of the thin walled shell so that

$$r = Xe_1 + Ye_2 + Ze_3 \tag{2.1}$$

where e_1, e_2, e_3 are unit vectors in the X, Y, Z directions respectively. Let the middle surface be defined by orthogonal curvilinear coordinates ξ_1', ξ_2', see Fig. 1, so that

$$X = X(\xi_1', \xi_2'), \quad Y = Y(\xi_1', \xi_2'), \quad Z = Z(\xi_1', \xi_2'). \tag{2.2}$$

Unit vectors t_1', t_2' tangent respectively to the ξ_1', ξ_2' directions are given by

$$t_1' = \frac{1}{\alpha_1'}\frac{\partial r}{\partial \xi_1'}, \quad t_2' = \frac{1}{\alpha_2'}\frac{\partial r}{\partial \xi_2'} \tag{2.3}$$

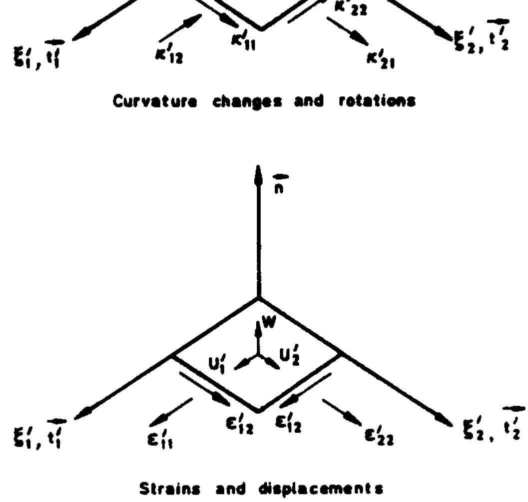

Fig. 1. Kinematic quantities in the orthogonal curvilinear coordinate system on the middle surface of shell triangular element.

where the coefficients α_1', α_2' of the first fundamental form are calculated from

$$\alpha_1' = \left|\frac{\partial r}{\partial \xi_1'}\right|, \quad \alpha_2' = \left|\frac{\partial r}{\partial \xi_2'}\right|. \tag{2.4}$$

The unit vector $\boldsymbol{n}$

$$\boldsymbol{n} = \boldsymbol{t}_1' \times \boldsymbol{t}_2' \tag{2.5}$$

is perpendicular to the plane of $\boldsymbol{t}_1'$ and $\boldsymbol{t}_2'$ with direction such that the orthogonal vectors $\boldsymbol{t}_1'$, $\boldsymbol{t}_2'$, $\boldsymbol{n}$ form a right handed system. The radii R_1', R_2' of curvature of the middle surface along lines of respectively constant ξ_2' and ξ_1' are given by

$$\frac{1}{R_1'} = -\frac{1}{\alpha_1'}\frac{\partial \boldsymbol{t}_1'}{\partial \xi_1'}\cdot\boldsymbol{n}, \quad \frac{1}{R_2'} = -\frac{1}{\alpha_2'}\frac{\partial \boldsymbol{t}_2'}{\partial \xi_2'}\cdot\boldsymbol{n} \tag{2.6}$$

while the radius R_{12}' of torsion is

$$\frac{1}{R_{12}'} = \frac{1}{\alpha_1'}\frac{\partial \boldsymbol{t}_2'}{\partial \xi_1'}\cdot\boldsymbol{n} = \frac{1}{\alpha_2'}\frac{\partial \boldsymbol{t}_1'}{\partial \xi_2'}\cdot\boldsymbol{n}. \tag{2.7}$$

If $1/R_{12}' = 0$ then the curvilinear coordinates ξ_1', ξ_2' follow the lines of principal curvature of the shell middle surface.

Let the displacement vector $\boldsymbol{U}$ be defined by

$$\boldsymbol{U} = U_X\boldsymbol{e}_1 + U_Y\boldsymbol{e}_2 + U_Z\boldsymbol{e}_3 \tag{2.8}$$

with

$$U_X = U_X(\xi_1', \xi_2'), \quad U_Y = U_Y(\xi_1', \xi_2'), \quad U_Z = U_Z(\xi_1', \xi_2'). \tag{2.9}$$

where U_X, U_Y, U_Z are components of displacement in the orthogonal X, Y, Z directions respectively. The displacement vector $\boldsymbol{U}$ is usually defined

$$\boldsymbol{U} = U_1'\boldsymbol{t}_1' + U_2'\boldsymbol{t}_2' + W\boldsymbol{n} \tag{2.10}$$

where U_1', U_2', W are respectively components of displacement in the ξ_1', ξ_2' and surface normal directions, these components may be calculated from

$$U_1' = \boldsymbol{U}\cdot\boldsymbol{t}_1', \quad U_2' = \boldsymbol{U}\cdot\boldsymbol{t}_2', \quad W = \boldsymbol{U}\cdot\boldsymbol{n}. \tag{2.11}$$

The components ϵ_{11}', ϵ_{22}', ϵ_{12}' of direct and shear strain in the ξ_1', ξ_2' coordinate system are

$$\left.\begin{aligned} \epsilon_{11}' &= \frac{1}{\alpha_1'}\frac{\partial \boldsymbol{U}}{\partial \xi_1'}\cdot\boldsymbol{t}_1', \quad \epsilon_{22}' = \frac{1}{\alpha_2'}\frac{\partial \boldsymbol{U}}{\partial \xi_2'}\cdot\boldsymbol{t}_2', \\ \epsilon_{12}' &= \frac{1}{2}\left(\frac{1}{\alpha_1'}\frac{\partial \boldsymbol{U}}{\partial \xi_1'}\cdot\boldsymbol{t}_2' + \frac{1}{\alpha_2'}\frac{\partial \boldsymbol{U}}{\partial \xi_2'}\cdot\boldsymbol{t}_1'\right). \end{aligned}\right\} \tag{2.12}$$

The rotation vector $\boldsymbol{\Phi}$ is introduced by

$$\boldsymbol{\Phi} = -\phi_2'\boldsymbol{t}_1' + \phi_1'\boldsymbol{t}_2' + \phi_n\boldsymbol{n} \tag{2.13}$$

with

$$\phi_1' = -\frac{1}{\alpha_1'}\frac{\partial \boldsymbol{U}}{\partial \xi_1'}\cdot\boldsymbol{n}, \quad \phi_2' = -\frac{1}{\alpha_2'}\frac{\partial \boldsymbol{U}}{\partial \xi_2'}\cdot\boldsymbol{n} \tag{2.14}$$

and with rotation ϕ_n about the surface normal

$$\phi_n = \frac{1}{2}\left(\frac{1}{\alpha_1'}\frac{\partial U}{\partial \xi_1'}\cdot t_2' - \frac{1}{\alpha_2'}\frac{\partial U}{\partial \xi_2'}\cdot t_1'\right). \tag{2.15}$$

The components of curvature change are given by

$$\left.\begin{aligned} -\kappa_{12}' &= \frac{1}{\alpha_1'}\frac{\partial \Phi}{\partial \xi_1'}\cdot t_1', \quad \kappa_{11}' = \frac{1}{\alpha_1'}\frac{\partial \Phi}{\partial \xi_1'}\cdot t_2' \\ -\kappa_{22}' &= \frac{1}{\alpha_2'}\frac{\partial \Phi}{\partial \xi_2'}\cdot t_1', \quad \kappa_{21}' = \frac{1}{\alpha_2'}\frac{\partial \Phi}{\partial \xi_2'}\cdot t_2'. \end{aligned}\right\} \tag{2.16}$$

When the strains ϵ_{11}', ϵ_{22}', ϵ_{12}' in the shell middle surface are all zero then the equations of compatibility provide the following relationships

$$\left.\begin{aligned} &\frac{\partial}{\partial \xi_2'}(\alpha_1'\kappa_{11}') - \frac{\partial}{\partial \xi_1'}(\alpha_2'\kappa_{21}') - \kappa_{22}'\frac{\partial \alpha_1'}{\partial \xi_2'} - \kappa_{12}'\frac{\partial \alpha_2'}{\partial \xi_1'} = 0, \\ &\frac{\partial}{\partial \xi_1'}(\alpha_2'\kappa_{22}') - \frac{\partial}{\partial \xi_2'}(\alpha_1'\kappa_{12}') - \kappa_{11}'\frac{\partial \alpha_2'}{\partial \xi_1'} - \kappa_{21}'\frac{\partial \alpha_1'}{\partial \xi_2'} = 0, \\ &\frac{\kappa_{11}'}{R_2'} + \frac{\kappa_{22}'}{R_1'} + \frac{1}{R_{12}'}(\kappa_{12}' + \kappa_{21}') = 0, \\ &\kappa_{12}' - \kappa_{21}' = 0, \end{aligned}\right\} \tag{2.17}$$

where κ_{11}', κ_{22}', κ_{12}', κ_{21}' are here the curvature changes of inextensional bending.

3. OBLIQUE CURVILINEAR COORDINATE SYSTEM

In the parametric representation employed in the sequel, the middle surface of the shell is defined by curvilinear coordinates ξ_1, ξ_2 which are no longer orthogonal. Thus,

$$X = X(\xi_1, \xi_2), \quad Y = Y(\xi_1, \xi_2), \quad Z = Z(\xi_1, \xi_2) \tag{3.1}$$

instead of eqn (2.2). It is therefore required to express the physical quantities of the orthogonal coordinate system ξ_1', ξ_2' in terms of these oblique coordinates.

Unit vectors t_1, t_2 tangent to the ξ_1, ξ_2 directions are given by

$$t_1 = \frac{1}{\alpha_1}\frac{\partial r}{\partial \xi_1}, \quad t_2 = \frac{1}{\alpha_2}\frac{\partial r}{\partial \xi_2} \tag{3.2}$$

where the coefficients of the first fundamental form are calculated from

$$\alpha_1 = \left|\frac{\partial r}{\partial \xi_1}\right|, \quad \alpha_2 = \left|\frac{\partial r}{\partial \xi_2}\right|. \tag{3.3}$$

These unit vectors include an angle β, see Fig. 2, where

$$\left.\begin{aligned} t_1\cdot t_2 &= \cos\beta, \quad 0 < \beta < \pi. \\ |t_1 \times t_2| &= \sin\beta. \end{aligned}\right\} \tag{3.4}$$

The vector t_1' of the orthogonal coordinate system is taken to include an angle λ with the vector t_1,

$$t_1'\cdot t_1 = \cos\lambda. \tag{3.5}$$

Since

$$\left.\begin{aligned} t_1'\cdot t_2 &= \cos\overline{\beta+\lambda}, \\ t_2'\cdot t_1 &= \sin\lambda, \\ t_2'\cdot t_2 &= \sin\overline{\beta+\lambda}, \end{aligned}\right\} \tag{3.6}$$

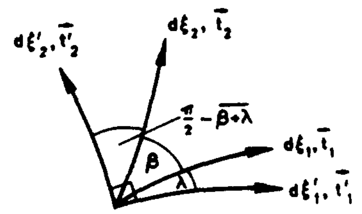

Fig. 2. Orthogonal and oblique coordinate systems.

see Fig. 2, it follows that

$$\left.\begin{aligned} t_1' &= \frac{1}{\sin\beta}(t_1 \sin\overline{\beta+\lambda} - t_2 \sin\lambda), \\ t_2' &= \frac{1}{\sin\beta}(-t_1 \cos\overline{\beta+\lambda} + t_2 \cos\lambda), \\ n &= \frac{1}{\sin\beta}(t_1 \times t_2) \end{aligned}\right\} \tag{3.7}$$

where n is the same vector as in eqn (2.5). It follows also that the rule for differentiation is

$$\begin{Bmatrix} \dfrac{1}{\alpha_1'}\dfrac{\partial}{\partial \xi_1'} \\ \\ \dfrac{1}{\alpha_2'}\dfrac{\partial}{\partial \xi_2'} \end{Bmatrix} = \frac{1}{\sin\beta}\begin{bmatrix} \sin\overline{\beta+\lambda} & -\sin\lambda \\ \\ -\cos\overline{\beta+\lambda} & \cos\lambda \end{bmatrix}\begin{Bmatrix} \dfrac{1}{\alpha_1}\dfrac{\partial}{\partial \xi_1} \\ \\ \dfrac{1}{\alpha_2}\dfrac{\partial}{\partial \xi_2} \end{Bmatrix} \tag{3.8}$$

Quantities ϵ_{11}, ϵ_{22}, ϵ_{12} in the oblique ξ_1, ξ_2 coordinate system are introduced by

$$\left.\begin{aligned} &\epsilon_{11} = \frac{1}{\alpha_1}\frac{\partial U}{\partial \xi_1}\cdot t_1, \quad \epsilon_{22} = \frac{1}{\alpha_2}\frac{\partial U}{\partial \xi_2}\cdot t_2, \\ &\epsilon_{12} = \frac{1}{2}\left(\frac{1}{\alpha_1}\frac{\partial U}{\partial \xi_1}\cdot t_2 + \frac{1}{\alpha_2}\frac{\partial U}{\partial \xi_2}\cdot t_1\right). \end{aligned}\right\} \tag{3.9}$$

These quantities may be termed "components of strain" since the strain of the middle surface is completely defined in terms of them; they are related to the orthogonal components ϵ_{11}', ϵ_{22}', ϵ_{12}' through

$$\left.\begin{aligned} \epsilon_1' &= \frac{1}{\sin^2\beta}\left(\epsilon_{11}\sin^2\overline{\beta+\lambda} + \epsilon_{22}\sin^2\lambda - 2\epsilon_{12}\sin\lambda\sin\overline{\beta+\lambda}\right) \\ \epsilon_{22}' &= \frac{1}{\sin^2\beta}(\epsilon_{11}\cos^2\overline{\beta+\lambda} + \epsilon_{22}\cos^2\lambda - 2\epsilon_{12}\cos\lambda\cos\overline{\beta+\lambda}) \\ \epsilon_{12}' &= \frac{1}{\sin^2\beta}\{-\epsilon_{11}\sin\overline{\beta+\lambda}\cos\overline{\beta+\lambda} - \epsilon_{22}\sin\lambda\cos\lambda \\ &\qquad + \epsilon_{12}(\sin\lambda\cos\overline{\beta+\lambda} + \sin\overline{\beta+\lambda}\cos\lambda)\} \end{aligned}\right\} \tag{3.10}$$

Note that ϵ_{11}, ϵ_{22} are true direct strains in the ξ_1, ξ_2 directions, e.g. with $\lambda = 0$, $\pi/2 - \beta$ respectively.

In like manner, quantities ϕ_1, ϕ_2 may be introduced by

$$\phi_1 = -\frac{1}{\sin\beta}\frac{1}{\alpha_1}\frac{\partial U}{\partial \xi_1}\cdot n, \quad \phi_2 = -\frac{1}{\sin\beta}\frac{1}{\alpha_2}\frac{\partial U}{\partial \xi_2}\cdot n \tag{3.11}$$

which are related to the orthogonal components ϕ_1', ϕ_2' by

$$\phi_1' = \phi_1 \sin \overline{\beta + \lambda} - \phi_2 \sin \lambda, \quad \phi_2' = -\phi_1 \cos \overline{\beta + \lambda} + \phi_2 \cos \lambda. \tag{3.12}$$

The rotation ϕ_n of eqn (2.15) may alternatively be expressed by

$$\phi_n = \frac{1}{2 \sin \beta}\left(\frac{1}{\alpha_1}\frac{\partial U}{\partial \xi_1}\cdot t_2 - \frac{1}{\alpha_2}\frac{\partial U}{\partial \xi_2}\cdot t_1\right) \tag{3.13}$$

and the rotation vector $\mathbf{\Phi}$ of eqn (2.13) by

$$\mathbf{\Phi} = -\phi_2 t_1 + \phi_1 t_2 + \phi_n n. \tag{3.14}$$

Quantities κ_{11}, κ_{22}, κ_{12}, κ_{21} are introduced by

$$\left.\begin{aligned} -\kappa_{12} &= \frac{1}{\alpha_1}\frac{\partial \mathbf{\Phi}}{\partial \xi_1}\cdot t_1, \quad \kappa_{11} = \frac{1}{\alpha_1}\frac{\partial \mathbf{\Phi}}{\partial \xi_1}\cdot t_2, \\ -\kappa_{22} &= \frac{1}{\alpha_2}\frac{\partial \mathbf{\Phi}}{\partial \xi_2}\cdot t_1, \quad \kappa_{21} = \frac{1}{\alpha_2}\frac{\partial \mathbf{\Phi}}{\partial \xi_2}\cdot t_2, \end{aligned}\right\} \tag{3.15}$$

which are related to the curvature changes of the orthogonal system through

$$\left.\begin{aligned} \kappa_{11}' &= \frac{1}{\sin^2\beta}(\kappa_{11}\cos\lambda\sin\overline{\beta+\lambda} - \kappa_{22}\sin\lambda\cos\overline{\beta+\lambda} \\ &\quad + \kappa_{12}\sin\overline{\beta+\lambda}\cos\overline{\beta+\lambda} - \kappa_{21}\sin\lambda\cos\lambda), \\ \kappa_{22}' &= \frac{1}{\sin^2\beta}(-\kappa_{11}\sin\lambda\cos\overline{\beta+\lambda} + \kappa_{22}\sin\overline{\beta+\lambda}\cos\lambda \\ &\quad - \kappa_{12}\sin\overline{\beta+\lambda}\cos\overline{\beta+\lambda} + \kappa_{21}\sin\lambda\cos\lambda), \\ \kappa_{12}' &= \frac{1}{\sin^2\beta}\{(\kappa_{11}-\kappa_{22})\sin\lambda\sin\overline{\beta+\lambda} + \kappa_{12}\sin^2\overline{\beta+\lambda} - \kappa_{21}\sin^2\lambda\}, \\ \kappa_{21}' &= \frac{1}{\sin^2\beta}\{-(\kappa_{11}-\kappa_{22})\cos\lambda\cos\overline{\beta+\lambda} - \kappa_{12}\cos^2\overline{\beta+\lambda} + \kappa_{21}\cos^2\lambda\}. \end{aligned}\right\} \tag{3.16}$$

The quantities κ_{11}, κ_{22}, κ_{12}, κ_{21} may be termed 'components of curvature change' since the curvature changes experienced by the middle surface are completely defined in terms of them; note in contrast with eqn (3.10), that κ_{11}, κ_{22} are not true curvature changes in the ξ_1, ξ_2 directions.

4. EQUATIONS OF THE SHELL TRIANGULAR ELEMENT

Consideration is now given to doubly curved shell triangular elements as described in the Introduction and where the middle surface is parametrically represented by quadratic polynomials in the oblique curvilinear coordinates ξ_1, ξ_2. The six noded triangular element is depicted in Fig. 3 with all three vertices resting on the OXY plane; the origin of rectangular cartesian coordinates X, Y, Z is located at node 5 with node 1 resting on the OX axis. The element class is thus defined by the six nodal coordinates X_1, X_3, Y_3, Z_2, Z_4, Z_6 with

$$\left.\begin{aligned} X_2 = \tfrac{1}{2}(X_1 + X_3), \quad Y_2 = Y_4 = \tfrac{1}{2}Y_3, \\ X_4 = \tfrac{1}{2}X_3, \quad X_5 = Y_1 = Y_5 = Y_6 = Z_1 = Z_3 = Z_5 = 0, \\ X_6 = \tfrac{1}{2}X_1. \end{aligned}\right\} \tag{4.1}$$

Note, however, that both method and form of solution remain unchanged when the element class is widened with $Z_1, Z_3, Z_5 \neq 0$.

The oblique curvilinear coordinates ξ_1, ξ_2 are shown in Fig. 4 with origin at node 5 so that,

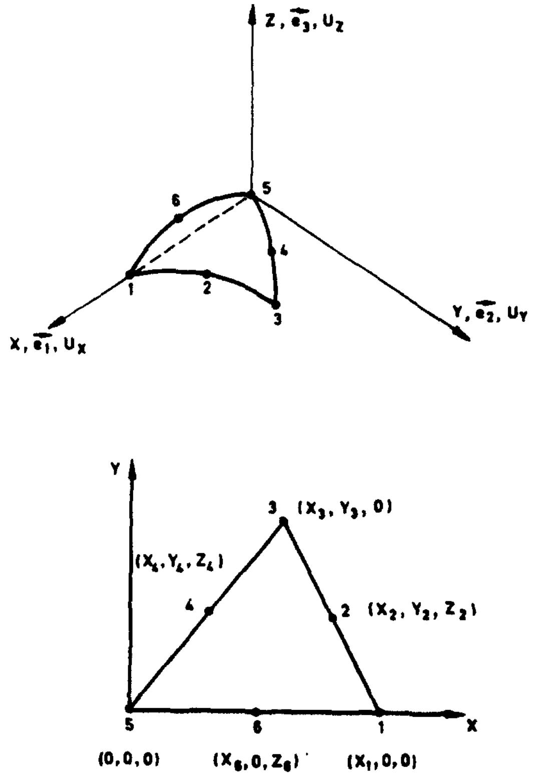

Fig. 3. Shell triangular element with vertices resting on OXY plane.

on the surface of the triangular element,

$$\left.\begin{aligned} &\text{side 1 is defined by } \xi_1+\xi_2=1,\\ &\text{side 2 is defined by } \xi_1=0,\\ &\text{side 3 is defined by } \xi_2=0. \end{aligned}\right\} \qquad (4.2)$$

With nodes located at coordinate positions defined by eqn (4.1), the quadratic representation requires that

$$\left.\begin{aligned} X &= X_1\xi_1+X_3\xi_2,\\ Y &= Y_3\xi_2,\\ Z &= 4\{Z_2\xi_1\xi_2+Z_4\xi_2(1-\xi_1-\xi_2)+Z_6\xi_1(1-\xi_1-\xi_2)\} \end{aligned}\right\} \qquad (4.3)$$

and the vector eqn (2.1), which describes the middle surface of the shell element, may now be

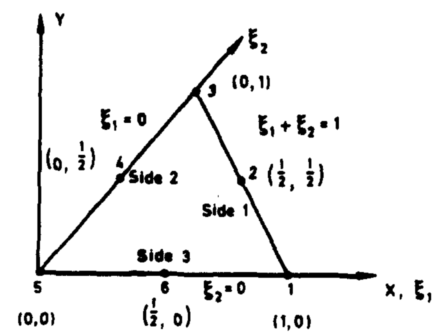

Fig. 4. Curvilinear coordinates of the quadratic parametric representation.

written

$$r = (X_1\xi_1 + X_3\xi_2)e_1 + Y_3\xi_2 e_2 + 4\{Z_2\xi_1\xi_2 + Z_4\xi_2(1-\xi_1-\xi_2) + Z_6\xi_1(1-\xi_1-\xi_2)\}e_3. \quad (4.4)$$

Unit vectors t_1, t_2, tangent respectively to the ξ_1, ξ_2 directions, are determined by eqn (3.2),

$$\left.\begin{aligned} t_1 &= \frac{1}{\alpha_1}\frac{\partial r}{\partial \xi_1} = \frac{1}{\alpha_1}[X_1 e_1 + 4\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\}e_3], \\ t_2 &= \frac{1}{\alpha_2}\frac{\partial r}{\partial \xi_2} = \frac{1}{\alpha_2}[X_3 e_1 + Y_3 e_2 + 4\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}e_3], \end{aligned}\right\} \quad (4.5)$$

with

$$\left.\begin{aligned} \alpha_1{}^2 &= X_1{}^2 + 16\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\}^2, \\ \alpha_2{}^2 &= X_3{}^2 + Y_3{}^2 + 16\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}^2; \end{aligned}\right\} \quad (4.6)$$

the trigonometric term $\cos\beta$ of eqn (3.4) is given by

$$\begin{aligned} \cos\beta &= t_1 \cdot t_2 \\ &= \frac{1}{\alpha_1\alpha_2}[X_1X_3 + 16\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\}\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}]. \end{aligned} \quad (4.7)$$

In the sequel, it is required to evaluate physical quantities in orthogonal directions at the sides of the triangular element. For this purpose, and typically for side 1, introduce new orthogonal cartesian coordinates X', Y' lying in the OXY plane

$$\left.\begin{aligned} X' &= \{X - \tfrac{1}{2}(X_1 + X_3)\}L' + \{Y - \tfrac{1}{2}(Y_1 + Y_3)\}M', \\ Y' &= -\{X - \tfrac{1}{2}(X_1 + X_3)\}M' + \{Y - \tfrac{1}{2}(Y_1 + Y_3)\}L' \end{aligned}\right\} \quad (4.8)$$

where, as in Fig. 5, the origin of the new coordinates is at (X_2, Y_2, 0) and OY' lies along the straight line joining nodes 1 and 3; the direction cosines are

$$L' = -\frac{(Y_1 - Y_3)}{l'}, \quad M' = \frac{(X_1 - X_3)}{l'} \quad (4.9)$$

with distance l' between nodes 1 and 3 calculated from

$$l'^2 = (X_1 - X_3)^2 + (Y_1 - Y_3)^2. \quad (4.10)$$

It is necessary to introduce also orthogonal unit vectors t_2' tangential to the curvilinear coordinate ξ_2' along the side and t_1' outwards pointing, as shown in Fig. 5, both vectors also

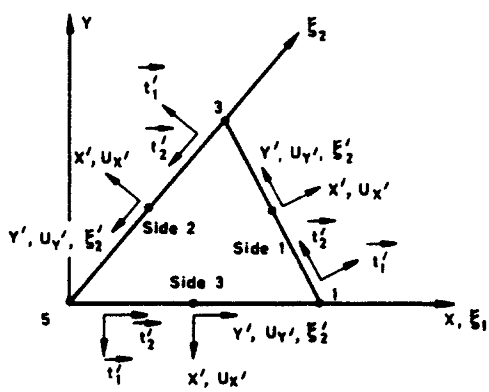

Fig. 5. Notation along sides of the shell triangular element.

tangential to the shell surface. The origin of ξ_2' is at the side node 2 with the vertices located at $\xi_2' = \pm\frac{1}{2}$. The relationship with coordinates ξ_1, ξ_2 is

$$\left.\begin{aligned} &\text{along side 1} \quad \xi_1 = \tfrac{1}{2} - \xi_2', \quad \xi_2 = \tfrac{1}{2} + \xi_2', \\ &\text{along side 2} \quad \xi_1 = 0, \quad \xi_2 = \tfrac{1}{2} - \xi_2', \\ &\text{along side 3} \quad \xi_1 = \tfrac{1}{2} + \xi_2', \quad \xi_2 = 0. \end{aligned}\right\} \tag{4.11}$$

The typical side 1 is described in terms of the new coordinates by

$$r' = l'\xi_2' e_2' + Z_2(1 - 4\xi_2'^2)e_3 \tag{4.12}$$

where e_2' is unit vector parallel with the straight line joining nodes 1 and 3,

$$e_2' = -M'e_1 + L'e_2. \tag{4.13}$$

The unit tangent vector t_2' in the direction of ξ_2' is

$$t_2' = \frac{1}{\alpha_2'}(l'e_2' - 8Z_2\xi_2' e_3) \tag{4.14}$$

with coefficient of the first fundamental form

$$\alpha_2'^2 = l'^2 + 64Z_2^2\xi_2'^2. \tag{4.15}$$

Since

$$\frac{\partial r}{\partial \xi_2'} = \frac{\partial r}{\partial \xi_1}\frac{\partial \xi_1}{\partial \xi_2'} + \frac{\partial r}{\partial \xi_2}\frac{\partial \xi_2}{\partial \xi_2'} \tag{4.16}$$

it follows that

$$\begin{aligned} t_2' &= \frac{\alpha_1}{\alpha_2'}\frac{\partial \xi_1}{\partial \xi_2'}t_1 + \frac{\alpha_2}{\alpha_2'}\frac{\partial \xi_2}{\partial \xi_2'}t_2 \\ &= \frac{1}{\sin\beta}(-t_1\cos\overline{\beta + \lambda} + t_2\cos\lambda) \end{aligned} \tag{4.17}$$

from eqn (3.7). This determines the angle λ for the side; formulae useful in the evaluation of trigonometric quantities $\cos\overline{\beta + \lambda}$, $\cos\lambda$, $\sin\overline{\beta + \lambda}$, $\sin\lambda$ are listed in Table 1 for each side the triangular element. Components of displacement $U_{X'}$, $U_{Y'}$ in the orthogonal X', Y' directions are given by

$$\left.\begin{aligned} U_{X'} &= U_X L' + U_Y M', \\ U_{Y'} &= -U_X M' + U_Y L'. \end{aligned}\right\} \tag{4.18}$$

Table 1. Trigonometric functions involving the angle λ expressed for each side of the triangular element

Side	$\cos\overline{\beta+\lambda}$	$\cos\lambda$	$\sin\overline{\beta+\lambda}$	$\sin\lambda$	$\alpha_2'^2$
1	$\frac{\alpha_1}{\alpha_2'}\sin\beta$	$\frac{\alpha_2}{\alpha_2'}\sin\beta$	$\frac{1}{\alpha_2'}(-\alpha_1\cos\beta+\alpha_2)$	$\frac{1}{\alpha_2'}(-\alpha_1+\alpha_2\cos\beta)$	$\alpha_1^2-2\alpha_1\alpha_2\cos\beta+\alpha_2^2$
2	0	$-\sin\beta$	-1	$-\cos\beta$	α_2^2
3	$-\sin\beta$	0	$\cos\beta$	1	α_1^2

In this quadratic parametric representation, the strain components ϵ_{11}, ϵ_{22}, ϵ_{12} of eqn (2.9) may be written as follows for the displacement vector U of eqn (2.8),

$$\left.\begin{aligned}
\epsilon_{11} &= \frac{1}{\alpha_1}\frac{\partial U}{\partial \xi_1}\cdot t_1 = \frac{1}{\alpha_1{}^2}\left[X_1\frac{\partial U_X}{\partial \xi_1} + 4\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\}\frac{\partial U_Z}{\partial \xi_1}\right],\\
\epsilon_{22} &= \frac{1}{\alpha_2}\frac{\partial U}{\partial \xi_2}\cdot t_2 = \frac{1}{\alpha_2{}^2}\left[X_3\frac{\partial U_X}{\partial \xi_2} + Y_3\frac{\partial U_Y}{\partial \xi_2} + 4\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}\frac{\partial U_Z}{\partial \xi_2}\right],\\
\epsilon_{12} &= \frac{1}{2}\left(\frac{1}{\alpha_1}\frac{\partial U}{\partial \xi_1}\cdot t_2 + \frac{1}{\alpha_2}\frac{\partial U}{\partial \xi_2}\cdot t_1\right)\\
&= \frac{1}{2\alpha_1\alpha_2}\left[X_3\frac{\partial U_X}{\partial \xi_1} + Y_3\frac{\partial U_Y}{\partial \xi_1} + 4\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}\frac{\partial U_Z}{\partial \xi_1}\right.\\
&\quad \left. + X_1\frac{\partial U_X}{\partial \xi_2} + 4\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\}\frac{\partial U_Z}{\partial \xi_2}\right].
\end{aligned}\right\} \tag{4.19}$$

These strain components satisfy a relationship

$$\begin{aligned}
&\frac{\partial^2}{\partial \xi_1{}^2}(\alpha_2{}^2\epsilon_{22}) + \frac{\partial^2}{\partial \xi_2{}^2}(\alpha_1{}^2\epsilon_{11}) - \frac{\partial^2}{\partial \xi_1\partial \xi_2}(2\alpha_1\alpha_2\epsilon_{12})\\
&= 8\left\{Z_4\frac{\partial^2}{\partial \xi_1{}^2} + Z_6\frac{\partial^2}{\partial \xi_2{}^2} + (Z_2 - Z_4 - Z_6)\frac{\partial^2}{\partial \xi_1\partial \xi_2}\right\}U_Z
\end{aligned} \tag{4.20}$$

which involves only the displacement U_Z; this relationship is of significance in the sequel when solving for inextensional bending. Formulae are listed in Table 2 which give the resolution of these strains into orthogonal components ϵ'_{11}, ϵ'_{22}, ϵ'_{12} acting at each side of the triangular element, see Fig. 5; note in particular that the strain ϵ'_{22} acts in the direction of ξ'_2 along each side. When the displacements of an arbitrary rigid body movement,

$$\left.\begin{aligned}
U_X &= a_1 + a_2Y - a_5Z,\\
U_Y &= a_3 - a_2X - a_6Z,\\
U_Z &= a_4 + a_5X + a_6Y,
\end{aligned}\right\} \tag{4.21}$$

where $a_1, \ldots, a_6$ are constants, are substituted into eqns (4.19) then all three strain components are zero as is required; the representation of X, Y, Z in terms of the surface coordinates ξ_1, ξ_2 is given by eqn (4.3). The special displacement form

$$U_X = a_7X, \quad U_Y = a_7Y, \quad U_Z = a_7Z \tag{4.22}$$

is of interest because when substituted into eqns (4.19) it provides the strain condition

$$\epsilon_{11} = \epsilon_{22} = a_7, \quad \epsilon_{12} = a_7\cos\beta, \tag{4.23}$$

Table 2. Strain components ϵ'_{11}, ϵ'_{22}, ϵ'_{12} for each side of the triangular element as shown in Fig. 5

Side	ϵ'_{11}	ϵ'_{22}	ϵ'_{12}
1	$\frac{1}{\alpha_2'^2\sin^2\beta}\left[\epsilon_{11}(-\alpha_1\cos\beta + \alpha_2)^2 + \epsilon_{22}(-\alpha_1 + \alpha_2\cos\beta)^2 - 2\epsilon_{12}\left\{\left(\alpha_1^2 + \alpha_2^2\right)\cos\beta - \alpha_1\alpha_2(1 + \cos^2\beta)\right\}\right]$	$\frac{1}{\alpha_2'^2}\left(\epsilon_{11}\alpha_1^2 + \epsilon_{22}\alpha_2^2 - 2\epsilon_{12}\alpha_1\alpha_2\right)$	$\frac{1}{\alpha_2'^2\sin\beta}\left\{\epsilon_{11}\alpha_1(\alpha_1\cos\beta - \alpha_2) + \epsilon_{22}\alpha_2(\alpha_1 - \alpha_2\cos\beta) - \epsilon_{12}\left(\alpha_1^2 - \alpha_2^2\right)\right\}$
2	$\frac{1}{\sin^2\beta}\left(\epsilon_{11} + \epsilon_{22}\cos^2\beta - 2\epsilon_{12}\cos\beta\right)$	ϵ_{22}	$\frac{1}{\sin\beta}(-\epsilon_{22}\cos\beta + \epsilon_{12})$
3	$\frac{1}{\sin^2\beta}\left(\epsilon_{11}\cos^2\beta + \epsilon_{22} - 2\epsilon_{12}\cos\beta\right)$	ϵ_{11}	$\frac{1}{\sin\beta}(\epsilon_{11}\cos\beta - \epsilon_{12})$

where use is made of eqns (4.5)–(4.7). Substitution of eqn (4.23) into Table 2 then shows that

$$\epsilon'_{11} = \epsilon'_{22} = a_7, \quad \epsilon'_{12} = 0; \tag{4.24}$$

this constant strain condition is accompanied by zero rotations ϕ'_1, ϕ'_2, ϕ_n as well as by zero curvature changes κ'_{11}, κ'_{22}, κ'_{12}, κ'_{21}, it is independent of the angle λ.

It is impracticable to write down explicit expressions for the curvature change components κ_{11}, κ_{22}, κ_{12}, κ_{21}, see eqn (3.15), in this quadratic representation. Instead, a simpler concept of arc curvature change, denoted κ'_{22A}, is introduced and used in the sequel to assess the quality of curvature change deformation. Typically for side 1, the vector eqn (4.12) describes a plane quadratic arc with unit tangent vector t'_2 given by eqn (4.14). The unit vector which lies in the plane of this arc, the osculating plane, and which is normal to t'_2 is

$$n_A = -\frac{dt'_2}{d\xi'_2}\bigg/\left|\frac{dt'_2}{d\xi'_2}\right| = \frac{1}{\alpha'_2}(8Z_2\xi'_2 e'_2 + l_{13}e_3). \tag{4.25}$$

The sense of this vector n_A corresponds with that of n as given by eqns (2.5) and (3.7). The vector equation for displacements in the plane of the arc is

$$U_A = U_{Y'}e'_2 + U_Z e_3 \tag{4.26}$$

where $U_{Y'}$ is expressed in terms of U_X and U_Y by eqn (4.18); the rotation ϕ'_{2A} of the arc in its own plane is

$$\phi'_{2A} = -\frac{1}{\alpha'_2}\frac{dU_A}{d\xi'_2}\cdot n_A = -\frac{1}{\alpha'^2_2}\left(8Z_2\xi'_2\frac{dU_{Y'}}{d\xi'_2} + l'\frac{dU_Z}{d\xi'_2}\right), \tag{4.27}$$

see eqn (2.14). The arc curvature change κ'_{22A} is therefore

$$\kappa'_{22A} = \frac{1}{\alpha'_2}\frac{d\phi'_{2A}}{d\xi'_2}$$

$$= \frac{128Z_2^2\xi'_2}{\alpha'^5_2}\left(8Z_2\xi'_2\frac{dU_{Y'}}{d\xi'_2} + l'\frac{dU_Z}{d\xi'_2}\right) - \frac{1}{\alpha'^3_2}\left(8Z_2\frac{dU_{Y'}}{d\xi'_2} + 8Z_2\xi'_2\frac{d^2U_{Y'}}{d\xi'^2_2} + l'\frac{d^2U_Z}{d\xi'^2_2}\right), \tag{4.28}$$

see eqn (2.16). Since

$$\epsilon'_{22} = \frac{1}{\alpha'^2_2}\left(l'\frac{dU_{Y'}}{d\xi'_2} - 8Z_2\xi'_2\frac{dU_Z}{d\xi'_2}\right), \tag{4.29}$$

the slope ϕ'_{2A0} for inextensional deformation $\epsilon'_{22} = 0$ is, on making use of eqn (4.15),

$$\phi'_{2A0} = -\frac{1}{l'}\frac{dU_Z}{d\xi'_2} \tag{4.30}$$

with corresponding arc inextensional curvature change

$$\kappa'_{22A0} = -\frac{1}{\alpha'_2 l'}\frac{d^2U_Z}{d\xi'^2_2}. \tag{4.31}$$

The coefficient α'_2 is nearly constant for a side with shallow arc, $\alpha'_2 \simeq l'$ see eqn (4.15), and κ'_{22A0} then tends to the same value of curvature change calculated according to Mushtari–Vlasov[10] shell theory. Care is required, however, if the concept of arc curvature change is used other than in a qualitative sense.

5. ISOPARAMETRIC REPRESENTATION OF THE DISPLACEMENTS

Suitable representation of the displacement components U_X, U_Y, U_Z of eqn (2.8) is important when considering satisfactory applications to the finite element method. If these components are represented isoparametrically by complete quadratics of the coordinates ξ_1, ξ_2,

$$\begin{Bmatrix} U_X \\ U_Y \\ U_Z \end{Bmatrix} = \sum_{i=1}^{6} \begin{Bmatrix} A_i \\ B_i \\ C_i \end{Bmatrix} N_i(\xi_1, \xi_2) \tag{5.1}$$

where $N_i(\xi_1, \xi_2)$ are shape functions defined as

$$\left.\begin{aligned} N_1 = \xi_1(2\xi_1 - 1), \quad & N_4 = 4\xi_2(1 - \xi_1 - \xi_2), \\ N_2 = 4\xi_1\xi_2, \quad & N_5 = (1 - \xi_1 - \xi_2)(1 - 2\xi_1 - 2\xi_2) \\ N_3 = \xi_2(2\xi_2 - 1), \quad & N_6 = 4\xi_1(1 - \xi_1 - \xi_2), \end{aligned}\right\} \tag{5.2}$$

then recovery of six arbitrary rigid body movements is admitted, eqn (4.21), as well as twelve separable states of combined strain and curvature change which include the special constant strain condition expressed by eqns (4.22)–(4.24). Note that independent states of constant strain ϵ'_{11}, ϵ'_{22}, ϵ'_{12} are not admitted by this representation, see eqn (4.19) and Table 2.

The present purpose, however, is to demonstrate that the quadratic isoparametric representation of eqn (5.1) does not lead to a satisfactory description of curvature change, independently of middle surface strain, for curved shell elements. Examine the kinematics in the plane of the arc formed typically by side 1, see Fig. 5. Components of displacement $U_{Y'}$, U_Z in this plane are represented by, see eqns (4.9), (4.18), (5.1),

$$\begin{Bmatrix} U_{Y'} \\ U_Z \end{Bmatrix} = \begin{bmatrix} -M' & L' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{Bmatrix} -\xi'_2(1 - 2\xi'_2) \\ 1 - 4\xi'^2_2 \\ \xi'_2(1 + 2\xi'_2) \end{Bmatrix}$$

$$= \begin{bmatrix} B'_1 & B'_2 & B'_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{Bmatrix} -\xi'_2(1 - 2\xi'_2) \\ 1 - 4\xi'^2_2 \\ \xi'_2(1 + 2\xi'_2) \end{Bmatrix}. \tag{5.3}$$

An equivalent form of representation, more convenient for the present purposes, is

$$\left.\begin{aligned} U_{Y'} &= b_1 + b_2 Y' + (b_3 - c_2 Z_2)\frac{Z}{Z_2}, \\ U_Z &= c_1 + c_2 Y' + (b_2 Z_2 + c_3)\frac{Z}{Z_2}, \end{aligned}\right\} \tag{5.4}$$

where, see eqn (4.11),

$$Y' = l'\xi'_2, \quad Z = Z_2(1 - 4\xi'^2_2). \tag{5.5}$$

Substitution of eqn (5.4) into eqn (4.29) gives

$$\epsilon'_{22} = b_2 - \frac{8b_3 l'\xi'_2}{\alpha'^2_2} + \frac{64c_3 Z_2 \xi'^2_2}{\alpha'^2_2}, \tag{5.6}$$

substitution into eqn (4.27) gives the in plane rotation ϕ'_{2A} of the arc as

$$\phi'_{2A} = \frac{64b_3 Z_2 \xi'^2_2}{\alpha'^2_2} - c_2 + \frac{8c_3 l\xi'_2}{\alpha'^2_2}, \tag{5.7}$$

while substitution into eqn (4.28) gives the arc curvature change as

$$\kappa'_{22A} = \frac{128b_3Z_2l'^2\xi'_2}{\alpha_2'^5} + \frac{8c_3l'(l'^2 - 64Z_2^2\xi_2'^2)}{\alpha_2'^5}. \tag{5.8}$$

Coefficients b_1, c_1, c_2 are seen to control the rigid body movement of the arc in its own plane, while the term with coefficient b_2 provides a constant value of the strain ϵ'_{22} along the side with zero rotation and curvature change, see eqns (4.22)–(4.24).

The term with coefficient b_3 describes a strain ϵ'_{22} which is antisymmetric in ξ'_2; this strain is accompanied with an arc curvature change κ'_{22A}, see eqn (5.8). Note here that a curvature change κ'_{22A} provides maximum fibre strain $\kappa'_{22A}h/2$ where h is the wall thickness of the shell element. For this term,

$$\frac{\kappa'_{22A}h/2}{\epsilon'_{22}} = -\frac{(128b_3Z_2l'^2\xi'_2/\alpha_2'^5)(h/2)}{8b_3l'\xi'_2/\alpha_2'^2} = -\frac{h}{R_A} \tag{5.9}$$

where R_A is the radius of curvature of the arc,

$$\frac{1}{R_A} = -\frac{1}{\alpha'_2}\frac{\mathrm{d}t'_2}{\mathrm{d}\xi'_2}\cdot n_A = \frac{8Z_2l'}{\alpha_2'^3} \tag{5.10}$$

on making substitutions from eqns (4.14) and (4.25). Now, because of the underlying Love-Kirchhoff assumptions in first approximation shell theory, it is not possible to calculate fibre strain (or stress) to within a relative accuracy closer than h/R, see Koiter[1], with R any radius of curvature or torsion of the reference surface. Equation (5.9) shows that the fibre strains from the accompanying (and unwanted) curvature change κ'_{22A} of the coefficient b_3 in eqn (5.8) are therefore of no consequence in the context of fibre strain accuracy.

This leaves available the term with coefficient c_3 to describe the arc curvature change κ'_{22A} but this is accompanied with unwanted middle surface strain, eqn (5.6), where

$$\frac{\epsilon'_{22}}{\kappa'_{22A}h/2} = \frac{16c_3Z_2/\alpha_2'^2}{(8c_3l'^3/\alpha_2'^5)h/2} \simeq \frac{4Z_2}{h} \tag{5.11}$$

with the physical quantities evaluated at appropriate values of ξ'_2. For

$$\frac{4Z_2}{h} \leq \frac{h}{R_A}, \tag{5.12}$$

see eqn (5.9), the angle $(8Z_2/R_A)^{1/2}$ which is approximately subtended by the arc is somewhat less than 0.33° for say $h = 0.04$ and $R_A = 10$. In a more relaxed criterion,

$$\frac{4Z_2}{h} \leq 0.1, \tag{5.13}$$

the arc subtended angle can increase to about 1.6° for the same values of h and R_A. The geometrical constraints are so severe that adequate description of independent curvature change for a curved shell element is seen to require a form of representation for the displacement components U_X, U_Y, U_Z which embodies inextensibility, $\epsilon'_{11} = \epsilon'_{22} = \epsilon'_{12} = 0$, of the middle surface.

6. INEXTENSIONAL BENDING IN TERMS OF CUBIC POLYNOMIALS

The components ϵ_{11}, ϵ_{22}, ϵ_{12} of eqn (3.9) completely define the state of middle surface strain throughout the shell element. Inextensibility is therefore governed by the three partial differential equations of first degree in U_X, U_Y, U_Z which are obtained from eqn (4.19) by enforcing

$$\alpha_1^2\epsilon_{11} = 0, \quad \alpha_2^2\epsilon_{22} = 0, \quad 2\alpha_1\alpha_2\epsilon_{12} = 0. \tag{6.1}$$

Solutions to these equations are sought in the form

$$\left.\begin{aligned} U_X &= \frac{16}{X_1}(a_2\xi_1 + a_3\xi_2 + a_4\xi_1^2 + a_5\xi_1\xi_2 + a_6\xi_2^2 \\ &\quad + a_7\xi_1^3 + a_8\xi_1^2\xi_2 + a_9\xi_1\xi_2^2 + a_{10}\xi_2^3), \\ U_Y &= -\frac{X_3}{Y_3}U_X + \frac{16}{Y_3}(b_3\xi_2 + b_4\xi_1^2 + b_5\xi_1\xi_2 + b_6\xi_2^2 \\ &\quad + b_7\xi_1^3 + b_8\xi_1^2\xi_2 + b_9\xi_1\xi_2^2 + b_{10}\xi_2^3), \\ U_Z &= 4C_2\xi_1\xi_2 + 4C_4\xi_2(1-\xi_1-\xi_2) + 4C_6\xi_1(1-\xi_1-\xi_2), \end{aligned}\right\} \tag{6.2}$$

where the U_Z displacement is set to zero at the vertices of the triangular element, *ie*

$$U_Z = 0 \quad \text{at nodes } 1, 3, 5 \tag{6.3}$$

see Figs. 3 and 4. An arbitrary rigid body movement parallel to the OXY plane,

$$\left.\begin{aligned} U_{XRB} &= \frac{16}{X_1}(a_1 - b_2\xi_2), \\ U_{YRB} &= -\frac{X_3}{Y_3}U_{XRB} + \frac{16}{Y_3}(b_1 + b_2\xi_1) \end{aligned}\right\} \tag{6.4}$$

see eqns (4.21) and (4.3), may be added to eqn (6.2) without affecting the curvature changes.

The twenty constants $a_2, a_3, \ldots, a_{10}$, $b_3, b_4, \ldots, b_{10}$ and C_2, C_4, C_6 are determined by substituting eqn (6.2) into eqn (4.19) to derive quadratic expressions for the strain quantities $\alpha_1^2\epsilon_{11}$, $\alpha_2^2\epsilon_{22}$ and $2\alpha_1\alpha_2\epsilon_{12}$; equating these expressions to zero, eqn (6.1), provides eighteen conditions and hence two inextensional bending solutions. The numerical results, given later, indicate that these solutions correspond to slowly varying values of the curvature changes for relatively shallow triangular elements such as are encountered in finite element analysis. A curved shell can support only two, ie not three, such separable inextensional bending behaviours because the compatibility condition, see eqn (2.17),

$$\frac{\kappa'_{11}}{R'_2} + \frac{\kappa'_{22}}{R'_1} + \frac{2\kappa'_{12}}{R'_{12}} = 0 \tag{6.5}$$

Table 3. Values of the physical quantities for specimen triangular element with positive Gaussian curvature

Side	Node	U_X	U_Y	U_Z	ϕ'_1	κ'_{11}	κ'_{22}	$\kappa'_{12}, \kappa'_{21}$	κ'_{22A0}
	1	0.5333	0	0	-5.752	-1.870	1.974	3.509	1.961
1	2	0.3000	-0.2887	1.000	-2.309	-1.987	2.013	3.464	2.000
	3	0.2667	-0.4619	0	1.132	-2.182	1.974	3.420	1.961
	3	0.2667	-0.4619	0	1.132	-2.182	1.974	-3.420	1.961
2	4	0.2333	-0.2887	1.000	-2.309	-1.987	2.013	-3.464	2.000
	5	0	0		-5.752	-1.870	1.974	-3.509	1.961
	5	0	0	0	4.620	4.051	-3.947	-0.0888	-3.922
3	6	0.2667	-0.2309	-2.000	4.619	3.974	-4.027	0	-4.000
	1	0.5333	0	0	4.620	4.051	-3.947	0.0888	-3.922

Side	Node	U_X	U_Y	U_Z	ϕ'_1	κ'_{11}	κ'_{22}	$\kappa'_{12}, \kappa'_{21}$	κ'_{22A0}
	1	0	-0.3079	0	-1.223	-2.078	1.974	-1.111	1.961
1	2	0.3000	-0.2887	1.000	-2.309	-1.987	2.013	-1.155	2.000
	3	0.8000	-0.1540	0	-3.397	-1.974	1.974	-1.199	1.961
	3	0.8000	-0.1540	0	3.397	1.974	-1.974	-1.199	-1.961
2	4	0.3000	-0.0192	-1.000	2.309	1.987	-2.013	-1.155	-2.000
	5	0	0		1.223	2.078	-1.974	-1.111	-1.961
	5	0	0	0	-2.355	0.1039	0	2.309	0
3	6	0	-0.1540	0	0	0	0	2.309	0
	1	0	-0.3079	0	2.355	-0.1039	0	2.309	0

Nodal coordinates, see Fig 3, which have non-zero values are $X_1 = 2$, $X_3 = 1$, $Y_3 = \sqrt{3}$, $Z_2 = Z_4 = Z_6 = 0.1$.

must be satisfied by curvature changes of inextensional bending, it is identically satisfied for flat plates where $Z_2 = Z_4 = Z_6 = 0$. This compatibility condition is reflected in the relationship given by eqn (4.20) where, on making use of eqn (6.1),

$$\left\{Z_4 \frac{\partial^2}{\partial \xi_1{}^2} + Z_6 \frac{\partial^2}{\partial \xi_2{}^2} + (Z_2 - Z_4 - Z_6) \frac{\partial}{\partial \xi_1 \partial \xi_2}\right\} U_Z = 0. \tag{6.6}$$

A substitution from the solution form given in eqn (6.2) shows that the constants C_2, C_4, C_6 must satisfy the condition

$$C_2(Z_2 - Z_4 - Z_6) + C_4(-Z_2 + Z_4 - Z_6) + C_6(-Z_2 - Z_4 + Z_6) = 0. \tag{6.7}$$

The remaining seventeen constants $a_2, a_3, \ldots, a_{10}$, $b_3, b_4, \ldots, b_{10}$ may now be determined by the procedure just described and expressed in terms of C_2, C_4, C_6 as follows

$$\left.\begin{array}{ll}
a_2 = -C_6 Z_6, & \\
a_3 = -(C_4 Z_6 + C_6 Z_4), & b_3 = -C_4 Z_4, \\
a_4 = -2a_2, & b_4 = -a_3, \\
a_5 = -C_2 Z_6 + C_4 Z_6 - C_6(Z_2 - Z_4 - 2Z_6), & b_5 = -C_2 Z_4 - C_4(Z_2 - 2Z_4 - Z_6) + C_6 Z_4, \\
a_6 = -a_3, & b_6 = -2b_3, \\
a_7 = 4a_2/3, & b_7 = -a_5/3, \\
a_8 = -a_5, & b_8 = 2a_3, \\
a_9 = 2a_3, & b_9 = -b_5, \\
a_{10} = -b_5/3, & b_{10} = 4b_3/3.
\end{array}\right\} \tag{6.8}$$

Numerical results for positive, zero and negative Gaussian curvature are presented respectively in Tables 3–5 for a specimen shell triangular element which projects onto the OXY plane as an equilateral triangle with side length 2 units. The tables list values of the displacement components U_X, U_Y, U_Z at all six nodes, see Fig. 5, as well as values of the rotation ϕ_i' and

Table 4. Values of the physical quantities for specimen triangular element with zero Gaussian curvature

Side	Node	U_X	U_Y	U_Z	ϕ_1'	κ_{11}'	κ_{22}'	$\kappa_{12}', \kappa_{21}'$	κ_{22A0}'
1	1	0	0	0	-1.132	0.6246	1.974	1.110	1.961
	2	0	-0.1540	1.000	0	0.6667	2.000	1.155	2.000
	3	0	-0.3079	0	1.132	0.6246	1.974	1.110	1.961
2	3	0	-0.3079	0	1.132	0.6246	1.974	-1.110	1.961
	4	0	-0.1540	1.000	0	0.6667	2.000	-1.155	2.000
	5	0	0	0	-1.132	0.6246	1.974	-1.110	1.961
3	5	0	0	0	2.309	2.598	0	0	0
	6	0	0	0	2.309	2.598	0	0	0
	1	0	0	0	2.309	2.598	0	0	0

Side	Node	U_X	U_Y	U_Z	ϕ_1'	κ_{11}'	κ_{22}'	$\kappa_{12}', \kappa_{21}'$	κ_{22A0}'
1	1	0	0	0	-1.132	-1.974	1.974	-1.199	1.961
	2	0.0667	-0.1155	1.000	-2.309	-2.000	2.000	-1.155	2.000
	3	0.5333	0	0	-3.397	-1.974	1.974	-1.199	1.961
2	3	0.5333	0	0	3.397	1.974	-1.974	-1.199	-1.961
	4	0.0667	0.115	-1.000	2.309	2.000	-2.000	-1.155	-2.000
	5	0	0	0	1.132	1.974	-1.974	-1.199	-1.961
3	5	0	0	0	-2.309	0	0	2.309	0
	6	0	0	0	0	0	0	2.309	0
	1	0	0	0	2.309	0	0	2.309	0

Nodal coordinates, see Fig 3, which have non-zero values are $X_1 = 2$, $X_3 = 1$, $Y_3 = \sqrt{3}$, $Z_2 = Z_4 = 0.1$.

Table 5. Values of the physical quantities for specimen triangular element with negative Gaussian curvature

Side	Node	U_X	U_Y	U_Z	ϕ_1'	κ_{11}'	κ_{22}'	$\kappa_{12}', \kappa_{21}'$	κ_{22A0}'
	1	0.1778	0	0	0.4680	1.461	2.071	0.2517	1.961
1	2	0.1000	-0.1989	1.000	0.7698	1.545	2.013	0.3849	2.000
	3	0.0889	-0.3592	0	1.132	1.560	1.974	0.3405	1.961
	3	0.0889	-0.3592	0	1.132	1.560	1.974	-0.3405	1.961
2	4	0.0778	-0.1989	1.000	0.7698	1.545	2.013	-0.3849	2.000
	5	0	0	0	0.4680	1.461	2.071	-0.2517	1.961
	5	0	0	0	1.600	2.152	1.381	-0.0888	1.307
3	6	0.0889	-0.0770	0.6667	1.540	2.100	1.411	0	1.333
	1	0.1778	0	0	1.600	2.152	1.381	0.0888	1.307

Side	Node	U_X	U_Y	U_Z	ϕ_1'	κ_{11}'	κ_{22}'	$\kappa_{12}', \kappa_{21}'$	κ_{22A0}'
	1	0	0.3079	0	-1.042	-1.774	2.071	-1.288	1.961
1	2	-0.1667	0.0577	1.000	-2.309	-1.987	2.013	-1.155	2.000
	3	0.2667	0.1540	0	-3.397	-0.974	1.974	-1.199	1.961
	3	0.2667	0.1540	0	3.397	1.974	-1.974	-1.199	-1.961
2	4	-0.1667	0.2502	1.000	2.309	1.987	-2.013	-1.155	-2.000
	5	0	0	0	1.042	1.774	-2.071	-1.288	-1.961
	5	0	0	0	-2.355	-0.2971	0	2.309	0
3	6	0	0.1540	0	0	0	0	2.309	0
	1	0	0.3079	0	2.355	0.2971	0	2.309	0

Nodal coordinates, see Fig 3, which have non-zero values are $X_1 = 2$, $X_3 = 1$, $Y_3 = \sqrt{3}$, $Z_2 = Z_4 = -Z_6 = 0.1$.

curvature changes κ_{11}', κ_{22}', $\kappa_{12}' = \kappa_{21}'$ at vertex and central nodes along each side of the triangle. These quantities refer to orthogonal coordinates ξ_1', ξ_2' where ξ_1' is the outwards pointing normal to the particular side. The arcs which form the curved sides have nearly constant radii of curvature with $R_A = 5$ and 5.3 units at the side and vertex nodes respectively, see eqn (5.10); these arcs subtend an angle of approx. 22.5° (in finite element analysis it is usually recommended that the element subtends an angle no greater than 10°). The last column in each Table lists values of the arc curvature change κ_{22A0}' of eqn (4.31) where the agreement to within 1% of the physical curvature change κ_{22}' is noteworthy for the elements with non-negative Gaussian curvature. This should not, however, be taken as a recommendation to employ arc curvature change other than qualitatively.

7. INEXTENSIONAL BENDING SOLUTIONS IN TERMS OF POLYNOMIALS OF ARBITRARY DEGREE

The main purpose of this paper was to develop inextensional bending solutions for U_X, U_Y, U_Z displacements in terms of low degree polynomials in the surface coordinates ξ_1 and ξ_2. It is, however, possible to write down exact closed form solutions in terms of homogeneous polynomials of arbitrary degree. Such solutions to eqns (6.1) are, for integer $n > 2$,

$$\left.\begin{aligned} U_X &= -\frac{4}{X_1}\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\}U_Z \\ &\quad + \frac{8}{nX_1}\left\{\frac{Z_4 c_{n-1}}{n-1} + (Z_2 - Z_4 - Z_6)c_n\right\}\xi_2^n - \frac{8Z_6}{X_1}\sum_{i=1}^{n}\frac{1}{n-i+1}c_i\xi_1^{n-i+1}\xi_2^{i-1}, \\ U_Y &= -\frac{X_3}{Y_3}U_X - \frac{4}{Y_3}\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}U_Z \\ &\quad + \frac{8}{nY_3}\left\{\frac{Z_6 c_2}{n-1} + (Z_2 - Z_4 - Z_6)c_1\right\}\xi_1^n - \frac{8Z_4}{Y_3}\sum_{i=1}^{n}\frac{1}{i}c_i\xi_1^{n-i}\xi_2^i, \\ U_Z &= \sum_{i=1}^{n} c_i\xi_1^{n-i}\xi_2^{i-1}, \end{aligned}\right\} \quad (7.1)$$

where the constants $c_1, c_2, \ldots, c_n$ are subject to $n-2$ conditions expressed by eqn (6.6) but are otherwise arbitrary.

For $n = 3$ the solutions of eqns (6.2) with (6.8) agree with eqn (7.1) apart from a rigid body movement.

When $n = 4$, for example, then

$$\left.\begin{aligned} U_X &= -\frac{4}{X_1}\left\{-2Z_6\xi_1 + (Z_2 - Z_4 - Z_6)\xi_2 + Z_6\right\}U_Z \\ &\quad + \frac{2}{X_1}\left\{\frac{Z_4 c_3}{3} + (Z_2 - Z_4 - Z_6)c_4\right\}\xi_2{}^4 - \frac{8Z_6}{X_1}\sum_{i=1}^{4}\frac{1}{5-i}c_i\xi_1^{5-i}\xi_2^{i-1}, \\ U_Y &= -\frac{X_3}{Y_3}U_X - \frac{4}{Y_3}\{(Z_2 - Z_4 - Z_6)\xi_1 - 2Z_4\xi_2 + Z_4\}U_Z \\ &\quad + \frac{2}{Y_3}\left\{\frac{Z_6 c_2}{3} + (Z_2 - Z_4 - Z_6)c_1\right\}\xi_1{}^4 - \frac{8Z_4}{Y_3}\sum_{i=1}^{4}\frac{1}{i}c_i\xi_1^{4-i}\xi_2^{i}, \\ U_Z &= c_1\xi_1{}^3 + c_2\xi_1{}^2\xi_2 + c_3\xi_1\xi_2{}^2 + c_4\xi_2{}^3, \end{aligned}\right\} \tag{7.2}$$

where eqn (6.6) requires that

$$\begin{aligned} 3Z_4c_1 + (Z_2 - Z_4 - Z_6)c_2 + Z_6c_3 &= 0, \\ Z_4c_2 + (Z_2 - Z_4 - Z_6)c_3 + 3Z_6c_4 &= 0. \end{aligned}$$

8. CONCLUSIONS

The strain/displacement and curvature change/displacement equations of first approximation shell theory are stated in terms of an arbitrary orthogonal system of curvilinear surface coordinates. These equations form the basis for evaluation of the physical quantities in any chosen direction and are then related to the equations of an arbitrary oblique curvilinear coordinate system.

Particular equations are derived for a class of shell triangular elements in quadratic parametric representation. Closed form polynomial exact solutions to the inextensional bending problem are presented with specimen numerical results; the cubic polynomial solutions correspond to curvature changes which, although not generally constant, are slowly varying for the shallow shell elements usually encountered in finite element analysis. The triangular element may have positive, zero or negative Gaussian curvature.

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